# Approximability of the Minimum Steiner Cycle Problem

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# Definitions

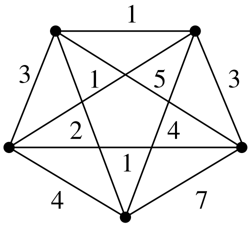
## Basic notions

- directed/undirected complete graph with *cost*$(\cdot)$
- complete distance network
- terminal

## Triangle inequality

$$cost(\overrightarrow{e}) \leq \beta \cdot (cost(\overrightarrow{a}) + cost(\overrightarrow{b}))$$

- *sharpened*: $\frac{1}{2} \leq \beta < 1$
- *metric*: $\beta = 1$
- *relaxed*: $\beta > 1$

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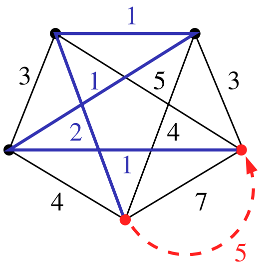

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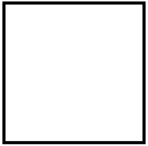

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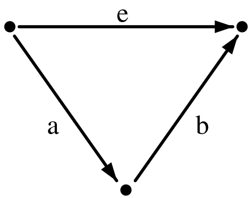
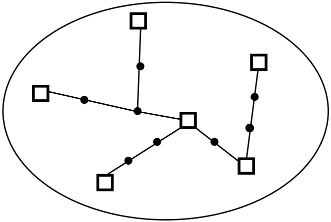


# Definitions

**Problems**

- minimum Steiner tree (MST)
- traveling salesman problem (TSP)



**minimum Steiner cycle problem (SCP)**

Find a *minimum-cost* cycle that contains every terminal *exactly once* (an other vertices at most once).

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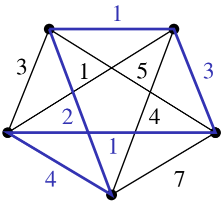


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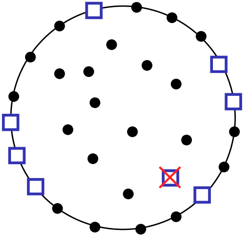

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# Characterization of SCP – 1

| | constant # □ | | non-constant # □ | |
|---|---|---|---|---|
| *cost* function | directed | undirected | directed | undirected |
| metric ($\beta = 1$) | ? | ? | | |
| relaxed ($\beta > 1$) | | | | |
| arbitrary | | | ? | ? |

# Characterization of SCP – 2

| | constant $\# \square$ | | non-constant $\# \square$ | |
|---|---|---|---|---|
| *cost* function | directed | undirected | directed | undirected |
| metric ($\beta = 1$) | P | P | | |
| relaxed ($\beta > 1$) | | | | |
| arbitrary | ? | ? | hard | hard |

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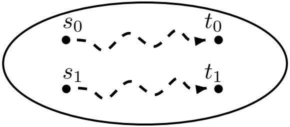
# Hardness of directed SCP with constant # terminals

**directed 2-vertex-disjoint path problem (D-2-VDPP)**

GIVEN: Directed graph $G$, distinct vertices $s_0$, $s_1$, $t_0$, $t_1$.

PROBLEM: Does $G$ contain *vertex disjoint* paths $s_0 \rightsquigarrow t_0$, $s_1 \rightsquigarrow t_1$?




**Theorem (Fortune, Hopcroft, Wyllie, 1980)**

The directed 2-vertex-disjoint path problem is NP-complete.

**Theorem**

Directed SCP with 4 terminals (D-4-SCP) has no $poly(|V|)$ approximation algorithm.

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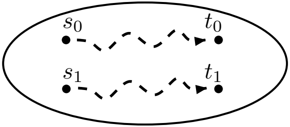


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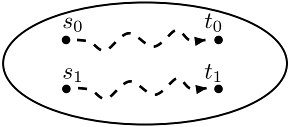


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**Theorem**

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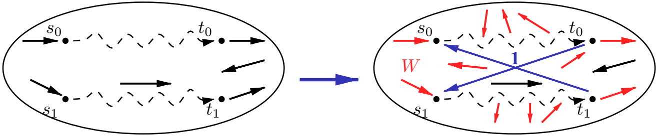
# Hardness of directed SCP with constant # terminals

## Proof

Proof by reduction to NP-complete D-2VDPP:

- suppose $\exists$ $p(|V|)$-approximation algorithm $\mathcal{A}$ for D-4-SCP
- for given graph $G$ for D-2VDPP, we construct graph $G'$ for D-4-SCP




- where $W > |V| \cdot p(|V|)$

# Hardness of directed SCP with constant # terminals

## Proof 2

Algorithm $\mathcal{A}$ applied on $G'$ gives cycle $C^*$ of $cost(C^*) < W$
$\Leftrightarrow$ $G$ contains 2 vertex-disjoint paths $s_0 \rightsquigarrow t_0$ and $s_1 \rightsquigarrow t_1$.

Proof:

$\Leftarrow$:
- pick cycle $C = (s_0 \rightsquigarrow t_0 \longrightarrow s_1 \rightsquigarrow t_1 \longrightarrow s_0)$ in $G'$
- cost of edges in $C$ is $1 \Rightarrow cost(C) \leq |V| \Rightarrow cost(C^*) \leq p(|V|)cost(C) < W$

$\Rightarrow$:
- since $cost(C^*) < W \Rightarrow$ each edge in $C^*$ must have cost 1
- order of vertices on $C^*$ must be $s_0, t_0, s_1, t_1 \Rightarrow$ we use $C^*$ to build two vertex-disjoint paths $s_0 \rightsquigarrow t_0$ and $s_1 \rightsquigarrow t_1$ in $G$

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# Characterization of SCP – 3

| | constant $\# \square$ | | non-constant $\# \square$ | |
|---|---|---|---|---|
| *cost* function | directed | undirected | directed | undirected |
| metric $(\beta = 1)$ | P | P | ? | ? |
| relaxed $(\beta > 1)$ | | | | |
| arbitrary | hard | ? | hard | hard |

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# Directed/undirected metric SCP

- $\beta = 1$: the direct edge is the cheapest connection of two vertices
- non-terminal vertices do not help us
- SCP = TSP on subgraph induced by terminals
- best known approximation algorithm for metric TSP:
  - directed graph – $\frac{2}{3}\log_2(|V|)$-approx.
  - undirected graph – $\frac{3}{2}$-approx.

# Characterization of SCP – 4

| | constant $\# \square$ | | non-constant $\# \square$ | |
|---|---|---|---|---|
| *cost* function | directed | undirected | directed | undirected |
| metric $(\beta = 1)$ | P | P | $\frac{2}{3}\log_2(\#\square)$ | $\frac{3}{2}$ |
| relaxed $(\beta > 1)$ | | ? | | ? |
| arbitrary | hard | ? | hard | hard |

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# SCP with relaxed triangle inequality – 1

## Idea of our algorithm

**Algorithm of Andreae, 2001**

INPUT: tree $T$ in a complete graph, $\beta$-relaxed triangle inequality
OUTPUT: Hamiltonian cycle $H$ on vertices from T such that $cost(H) \leq (\beta^2 + \beta) \cdot cost(T)$

Idea of our algorithm:

1. find a tree $T$ containing all terminals , such that $cost(T) \leq cost(OPT)$
2. plug $T$ into Andreae's algorithm to obtain $(\beta^2 + \beta)$-approximation of SCP

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# SCP with relaxed triangle inequality – 2

How to find $T$?

1. Build a complete distance network $D$ on terminals
2. Compute a minimum spanning tree $T_D$ on $D$
3. Build graph $G'$ by replacing the edges in $T_D$ by the corresponding shortest paths
4. Find a minimum spanning tree $T$ on $G'$

Cost of created objects:

$$cost(T) \leq cost(G') \leq cost(T_D) \leq cost(OPT)$$

# Summary

| | constant $\# \square$ | | non-constant $\# \square$ | |
|---|---|---|---|---|
| *cost* function | directed | undirected | directed | undirected |
| metric ($\beta = 1$) | $P$ | $P$ | $\frac{2}{3}\log_2(\#\square)$ | $\frac{3}{2}$ |
| relaxed ($\beta > 1$) | ? | $\beta^2 + \beta$ | ? | $\beta^2 + \beta$ |
| arbitrary | hard | ? | hard | hard |

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# Open problems

- How hard is the general undirected SCP with constant $\# \Box$?
- How can $\beta$-relaxed triangle inequality help?
- Does the fact that $\# \Box$ is constant help us?
- Is undirected SCP with constant $\# \Box$ and $\beta$-relaxed triangle inequality NP-hard?
- Can we improve the ratio $(\beta^2 + \beta)$?
- How hard is the directed case when $\# \Box$ is non-constant?

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Thank you for your attention!

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