Cop-Win Graphs with Maximal Capture-Time

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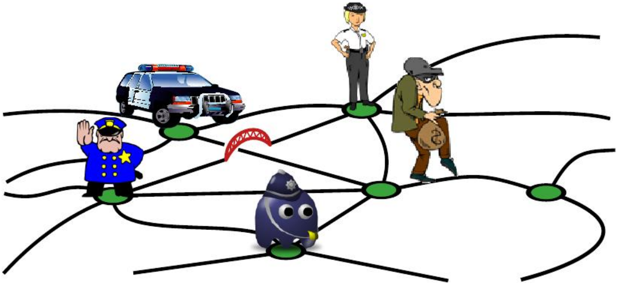


MEMICS 2009

The Game

Cop&robber is a two-player combinatorial game.

Played on a given undirected graph G.

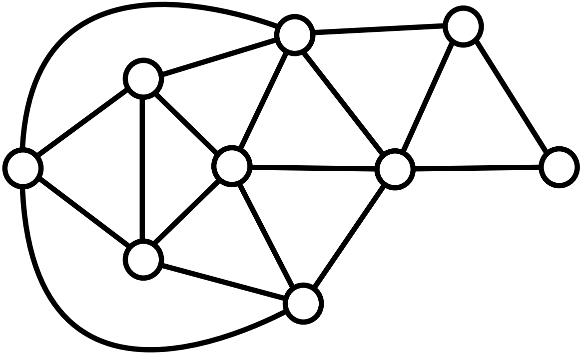
- The cop (first) and the robber choose their starting vertices.

- The players take turns:
- Cop moves to a neighbor vertex or passes
- Robbber moves to a neighbor vertex or passes

- The cop wins by catching the robber
- The robber wins by avoiding capture indefinitely

A sample game

Starring: the (**C**op), the (**R**obber)



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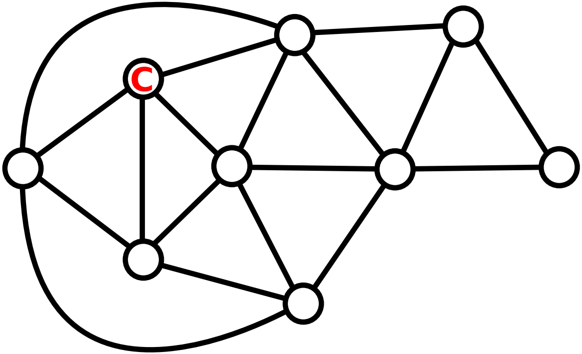

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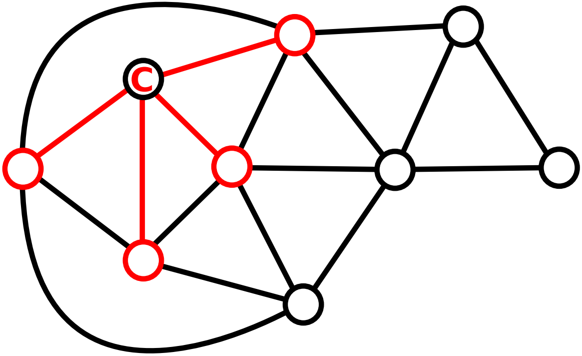

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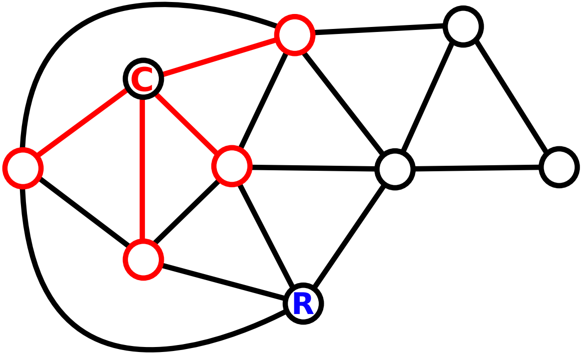

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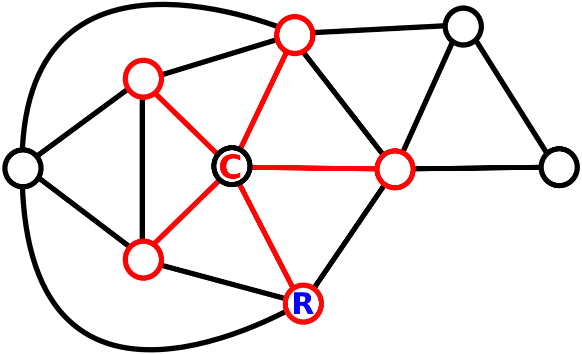

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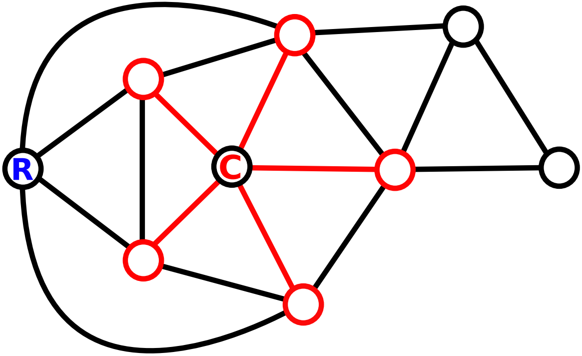

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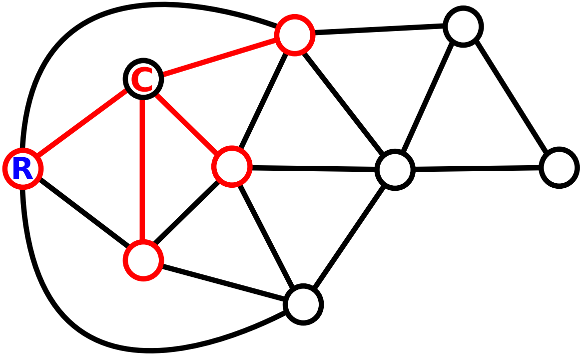

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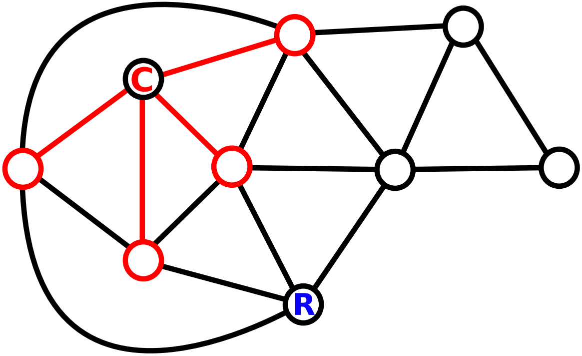

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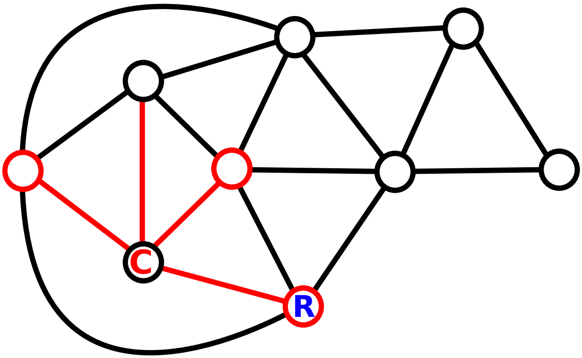

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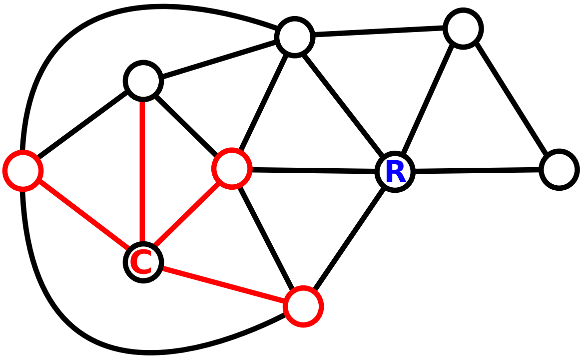

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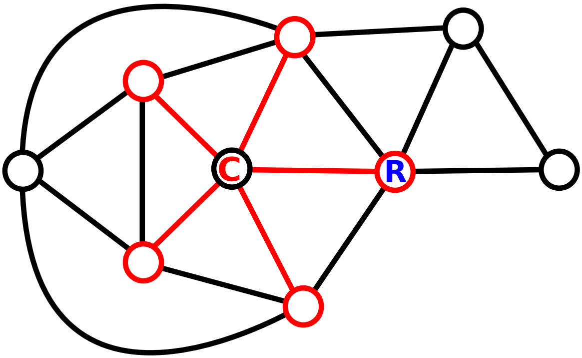

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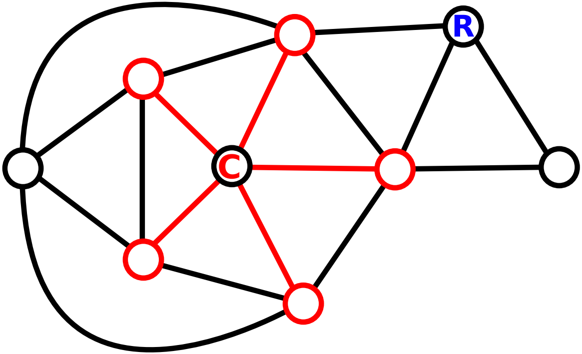

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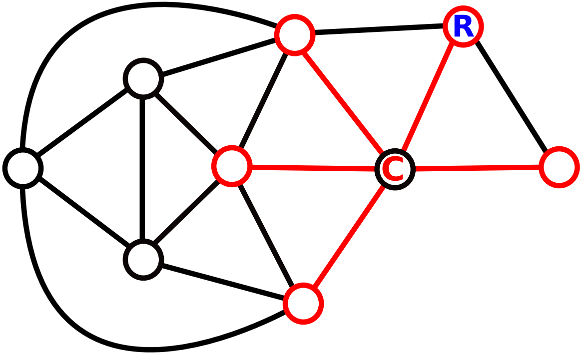

...and the robber is captured.

Facts & Definitions

Fact: On every graph, one of the players has a winning strategy. Graphs are either cop-win or robber-win.

Capture-time is the game length with both players playing optimally.

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Definition

The function $\mathtt{ct}_{\mathtt{max}}(n)$ is the maximal capture-time of a cop-win graph on n vertices.

Definition

Let $\mathcal{M}$ be the class of graphs having the maximal capture-time among all the graphs of the same size.

History

Nowakowski and Winkler (1983); Quilliot (1978, 1983)

Originally searching cave systems for moving targets. Considering the *number of searchers*.

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Relations to various *graph parameters*, definitions by games.

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Definition of *tree-width* by a cop&robber game with helicopters.

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Definition of *tree-width* by a cop&robber game with helicopters.

Recent focus on *capture-time* and *depth-parameters (tree-depth).*

Bonato, Golovach, Hahn, Kratochvíl (2006)

Bounds on the capture-time for various classes. For one-cop games $n - 4 \leq \mathtt{ct}_{\mathtt{max}}(n) \leq n - 3$.

Main results

We managed to:

Get the exact upper bound $\texttt{ct}_{\texttt{max}}(n)$.

Prove $\texttt{ct}_{\texttt{max}}(n) = n - 4$ and close the gap.

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Characterize the structure of the class $\mathcal{M}$.

Effective inductive construction of the entire class.

Show that $\mathcal{M}$ is exponentially big.

$\mathcal{M}$ contains at least 2^{n-8} graphs on n vertices.

The structure of cop-win graphs

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Vertex u dominates a vertex v if every neighbor of v (incl. v) is also a neighbor of u.

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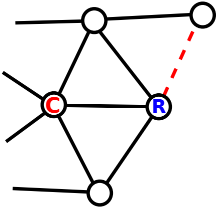
Observation

The robber can not be caught in a non-dominated vertex, the robber can always move out of cop's neighborhood.



Situation just before capture

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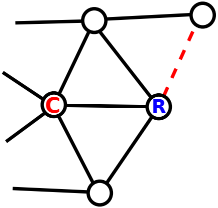

Situation just before capture

Characterization of cop-win graphs [Nowakowski, Winkler]

Graph G is cop-win iff G can be disassembled by consecutive removal of dominated vertices *(dismantlable)*.

The exact value of $\mathtt{ct}_{\mathtt{max}}$

Main Theorem

$\mathtt{ct}_{\mathtt{max}}(n) = n - 4$ for $n \geq 7$,
$\mathtt{ct}_{\mathtt{max}}(n) = \lfloor \frac{n}{2} \rfloor$ for $n \leq 7$.

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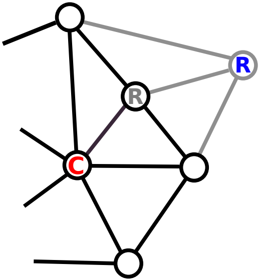
Upper bound on $\mathtt{ct}_{\mathtt{max}}$

- By removing a dominated vertex the capture-time drops by at most 1.



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- By removing a dominated vertex the capture-time drops by at most 1.
- From $\mathtt{ct}_{\mathtt{max}}(n) = k$ it follows that $\mathtt{ct}_{\mathtt{max}}(n+1) \leq k + 1$.
- It suffices to check that $\mathtt{ct}_{\mathtt{max}}(7) \leq 3$, but there are many cop-win graphs on 7 vertices ...

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- It suffices to check that $\mathtt{ct}_{\mathtt{max}}(7) \leq 3$, but there are many cop-win graphs on 7 vertices . . .
- Examine all cop-win graphs from $\mathcal{M}$ on 6 vertices. $\mathtt{ct}_{\mathtt{max}}(6) = 3 = n - 3$.
- None can be extended (by appending a dominated vertex) to have a higher capture-time.

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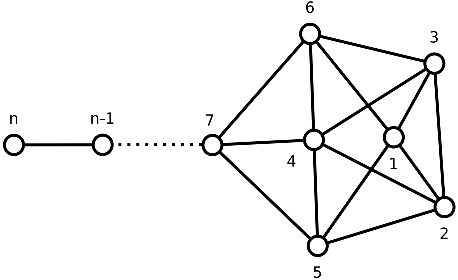
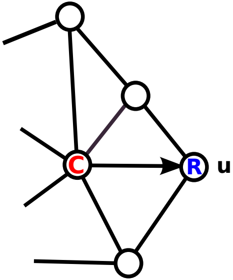
Lower bound on $\mathtt{ct}_{\mathtt{max}}$

- Paths are good examples for $n \leq 6$.
- For $n \geq 7$ we use graphs H_n:



We got simpler graphs than Bonato et. al.

The structure of $\mathcal{M}$



For all the graphs $G \in \mathcal{M}$ (on ≥ 8 vertices) we have:

- G is an extension of $G' \in \mathcal{M}$ by a dominated vertex.

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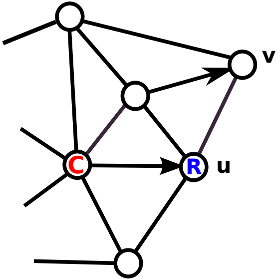

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- G has only one dominated vertex v. (otherwise it has a lower capture-time)

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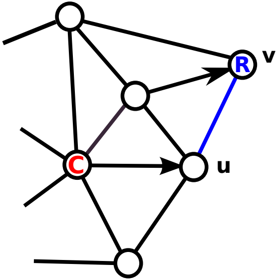

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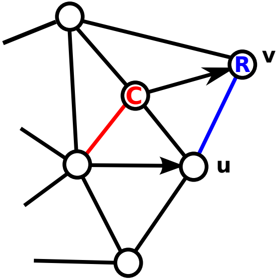

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- All vertices dominating v must be adjacent to vertices dominating u.
- G contains H_7 as an induced subgraph. (result of a case analysis on 8 vertices)

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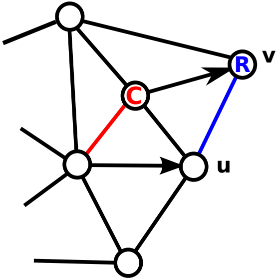

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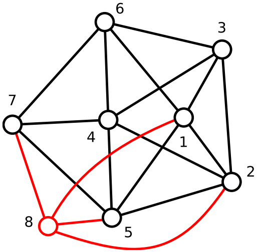
- All vertices dominating v must be adjacent to vertices dominating u.
- G contains H_7 as an induced subgraph.
- Therefore G contains H_n as a subgraph.

The exponential size of $\mathcal{M}$

Observation

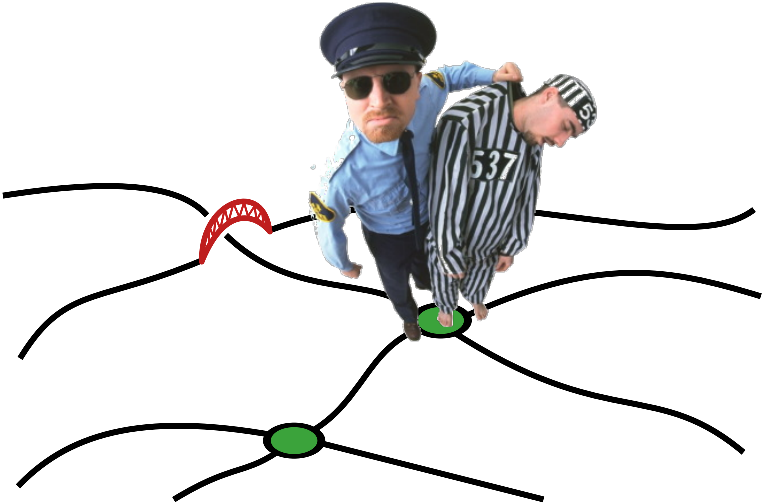
Every graph in $\mathcal{M}$ has unique copy of H_n (up to symmetry of H_7).



Sketch of an algorithm:

An asymmetric extension of H_7 as a base.
In every step choose neighbors from H_6.
Construction of 2^{n-8} graphs of $\mathcal{M}$ of size n.

The end?



Thank you for your attention!