The Parameterized Complexity of Oriented Colouring

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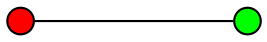
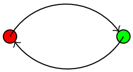
The problem: Oriented Chromatic Number (OCN)

- Introduced by Courcelle, studied by Nešetřil, Raspaud, Sopena and others
- Simple definition – We colour the vertices in a digraph so that:
  - neighbours always have different colours, and
  - all arcs between two colours have the same orientation
- Alternate definition – *homomorphism* of a digraph into a tournament
- Two related problems:
  - Decide oriented colourability by fixed number of colours (OCN_k)
  - Compute minimal number of colours (OCN)
- Applications in mobile networks

Oriented colouring examples





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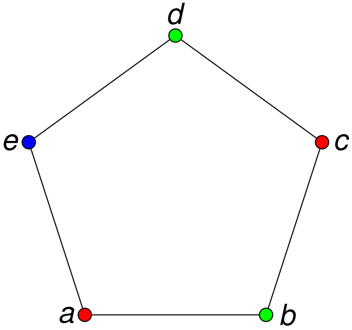

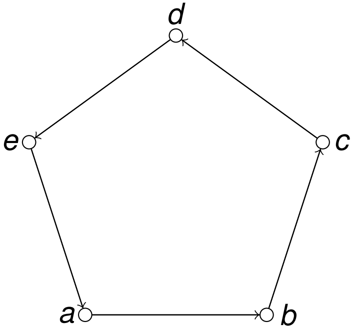

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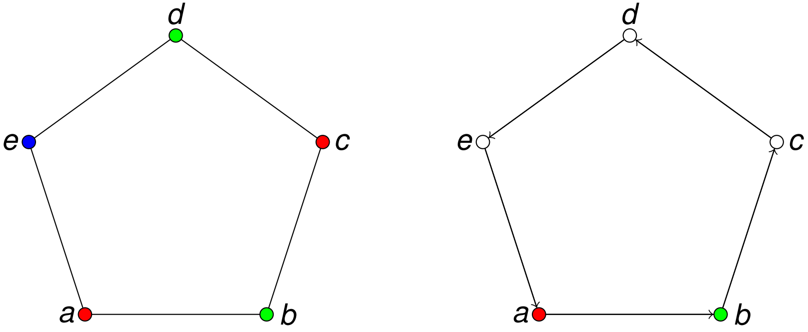

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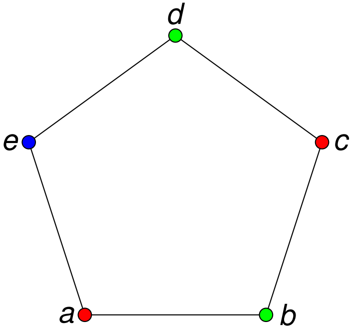

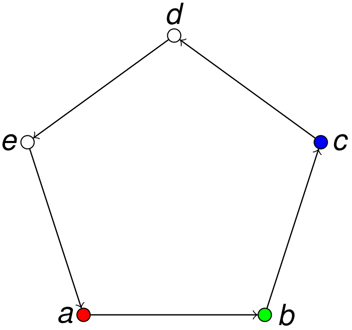

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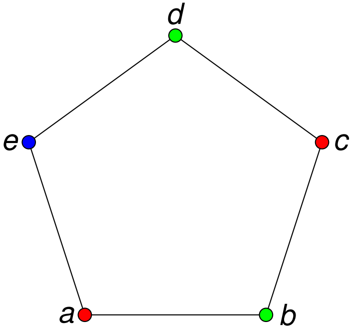

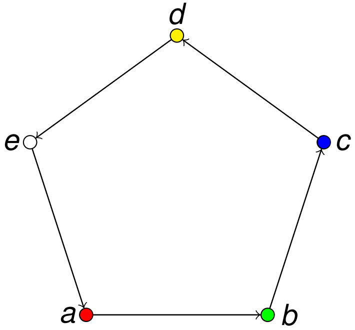

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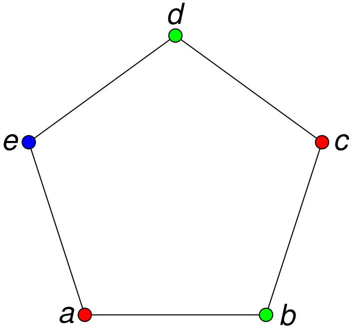

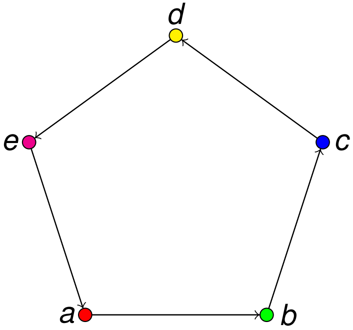

The tools: A crash course on Parameterized Algorithms

- Many interesting graph problems NP-hard on general graphs
- Input graphs in real-life applications usually specific in some way – often bounded structural parameter (bounded "width")
- This can be exploited by designing *parameterized algorithms* with a parameter t:
  - **FPT** algorithms are those running in time $O(poly(n) \cdot f(t))$
  - **XP** algorithms are those running in time $O(poly(n)^{f(t)})$

OCN and directed width parameters

- There exists a wide range of studied directed width parameters – D-width, DAG-width, Directed tree-width, Cycle rank, Kelly-width etc.
- Common feature: low, fixed value on DAGs (directed acyclic graphs)
- **Even deciding OCN_k is NP-hard on DAGs $\Longrightarrow$ "classical" directed width parameters useless for OCN**
- Bi-rank-width, a promising new directed width parameter, can decide OCN_k in FPT but computing OCN is open
- But we want to compute OCN...

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An undirected approach

- Most problems are studied separately on digraphs and undirected graphs
- However OCN only makes sense on digraphs
- Since directed width measures do not help, why not try using the undirected width measures of the underlying graph (“forgetting” edge orientations)?

Some successful undirected width parameters

Tree-width

- Robertson and Seymour (1984)
- Measures how close a graph is to being a tree
- Quite restrictive, but many NP-hard problems become FPT on graphs of bounded tree-width.

Rank-width

- Oum and Seymour (2004)
- Related to clique-width, but much better for algorithms
- Less restrictive than tree-width, but not as powerful – however still capable of solving many NP-hard problems in FPT or XP time.

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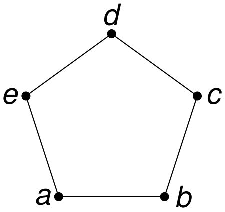
Tree-width and rank-width

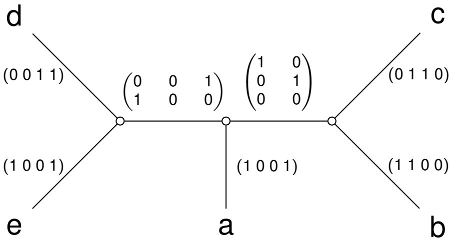
Similar scheme:

1. The tree/rank-width of G is the minimum width of a *tree/rank-decomposition* of G
2. The width of a tree/rank-decomposition is the maximum width of “something” in the decomposition

These decompositions are then used for algorithms as well.

An introduction to rank-width





OCN on digraphs of bounded rank-width

Unfortunately rank-width does not help with OCN:

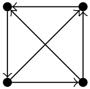
Theorem

Computing the oriented chromatic number of digraphs is DET-hard even when restricted to digraphs of bounded undirected rank- width

This is in contrast with undirected colouring – computable in XP on rank-width.

OCN on digraphs of bounded rank-width

Proof is based on a reduction from the isomorphism of tournaments (orientations of complete graphs)

- Tournament example: 
- Tournaments have rank-width 1 (their underlying graph is a complete graph)
- Tournament isomorphism proved to be DET-hard by Wagner (2007)
- Tournament isomorphism can be solved by computing OCN

OCN on digraphs of bounded rank-width

Sketch of reduction:

1. Create G as disjoint union of two possibly-isomorphic tournaments (each of size $\frac{|V(G)|}{2}$)
2. Compute OCN(G)
3. If $\mathrm{OCN}(G) = \frac{V(G)}{2}$ then the tournaments are isomorphic, otherwise they are not

OCN on digraphs of bounded tree-width

Theorem

OCN can be computed on digraphs of tree-width at most t in FPT time

- First parameterized algorithm for computing OCN
- Can be divided into three parts:
  - Undirected tree-width bounds OCN
  - Undirected tree-width bounds bi-rank-width
  - A FPT algorithm for deciding OCN_k on bi-rank-width

- Result of Hell and Nešetřil (2004) – does not give exact bounds
- Precise bounds can be obtained by combining results of Albertson, Chappell, Kierstead, Kndgen and Ramamurthi (2004) with results of Hell and Nešetřil
- Two-step process – Acyclic chromatic number

OCN on digraphs of bounded tree-width

Bounding bi-rank-width

- It is well known that tree-width bounds rank-width, but rank-width doesn't bound bi-rank-width
- New proof needed
- Core idea: Transform the tree-decomposition in a special way into a bi-rank-decomposition of the digraph
- Tree- and bi-rank-decompositions are not very similar – many technicalities

OCN on digraphs of bounded tree-width

Bounding bi-rank-width

- Rank-decomposition – each vertex stored in exactly one leaf
- Tree-decomposition – several vertices stored in nodes (*bags*) of decomposition, can repeat. All edges contained in some bag.

Bound on bi-rank-width comes from the nature of the tree-decomposition if transformation is done correctly.

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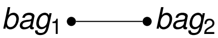

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OCN on digraphs of bounded tree-width

Deciding OCN_k on bi-rank-width

- Algorithm by Ganian, Hliněný, Kneis, Langer, Obdržálek, Rossmanith (2009)

Conclusions

- △ New approach to parameterized algorithms for problems restricted to directed graphs
- △ First parameterized algorithm for OCN
- △ Proof of tree-width bounding bi-rank-width is of independent interest
- ⋆ The FPT algorithm could be improved significantly with better bounds on OCN when tree-width is bounded
- ⋆ Computing OCN is still open on bi-rank-width

Thank
you
for
your
attention

$O(2^{b(t)^2+b(t)\cdot(t+1)(2t+3)} \cdot b(t)(2t+2)^3 \cdot \log b(t) \cdot |V(G)|)$, where
$b(t) = 2^{t+1} \cdot 2^{2^{t+1}-1}$