# Continuous-Time Stochastic Games with Time-Bounded Reachability

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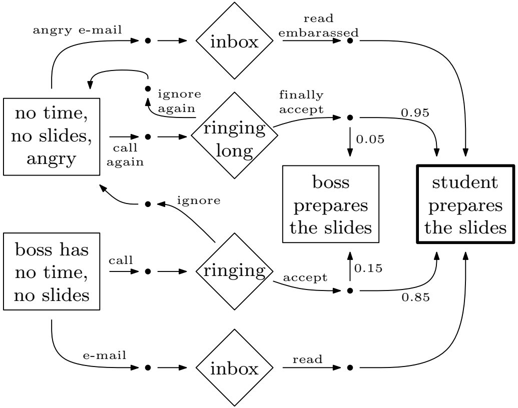
# Discrete-time stochastic game – example




- 2 players – □ and ◊
- in each vertex – a set of enabled *actions*
- for each action a probability distribution on the set of vertices

## goals of the players

- player □ – reach the target vertex
- player ◊ – avoid it

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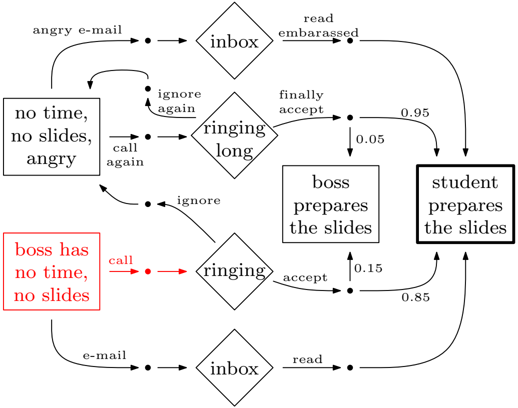


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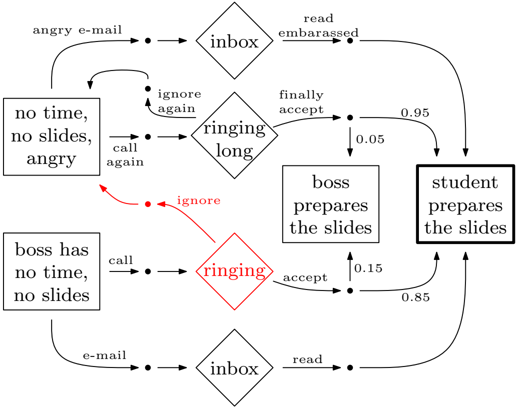


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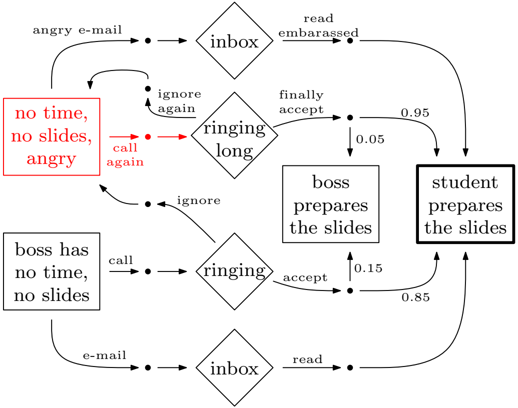


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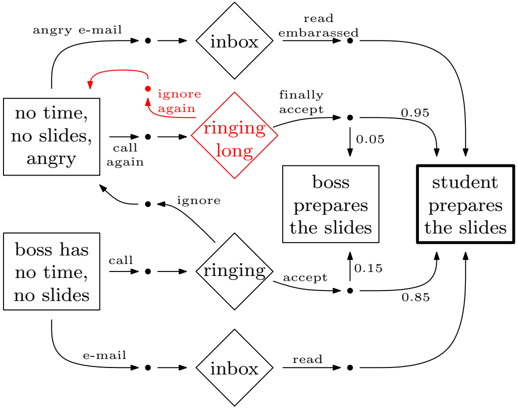


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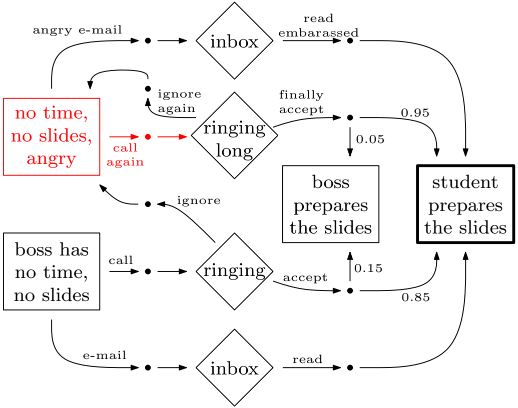


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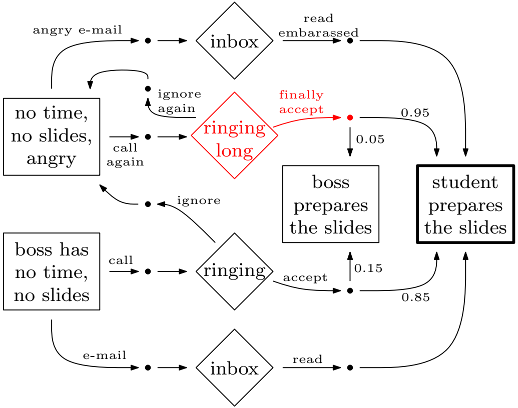


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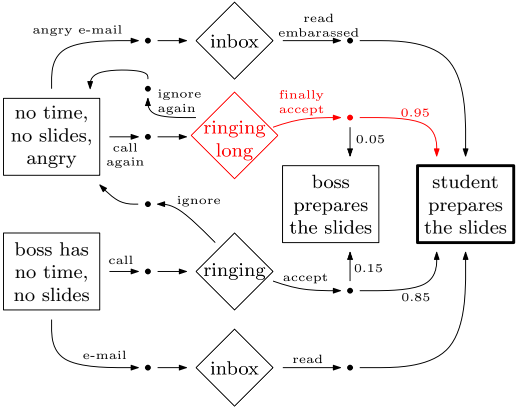


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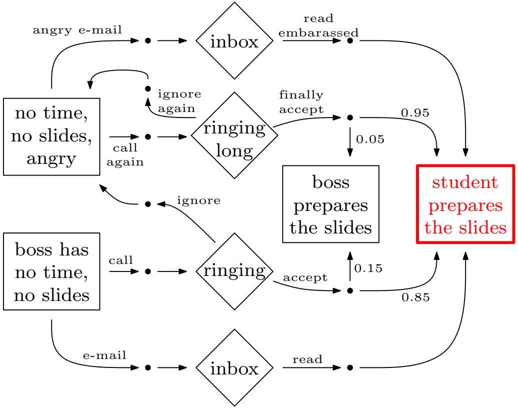


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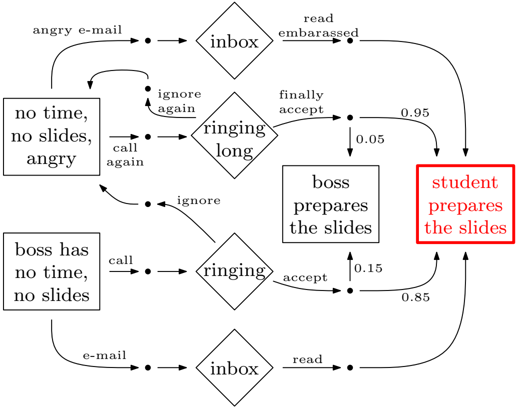


Probability that the player □ wins for strategies $\sigma$ and $\pi$ denoted by:
$\mathcal{P}^{\sigma,\pi}_{\text{boss-has-no-time}}(\mathit{Reach}(\{\text{student-prepares-slides}\}))$

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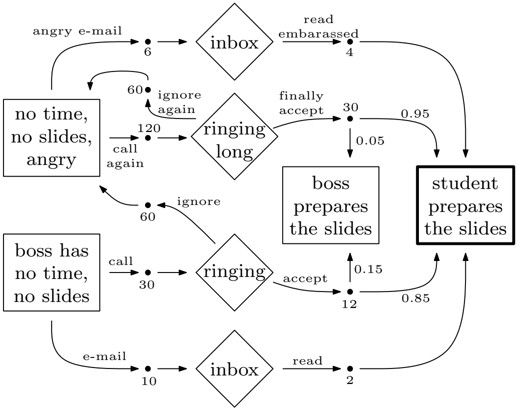
## now with time

the goal of the player □ – reach the target vertex in time 0.5




- performing an action takes time according to an exponential distribution
- each action has a rate - the parameter of the distribution
- rate ∼ how many such events occur in one time unit

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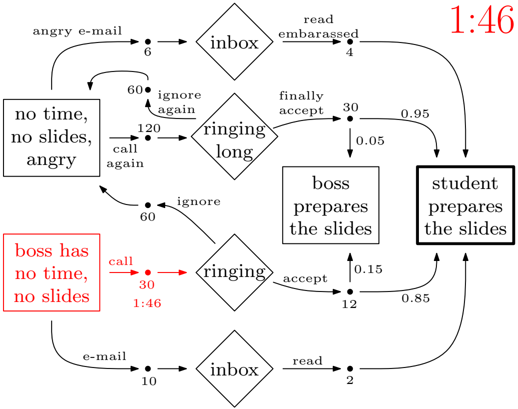


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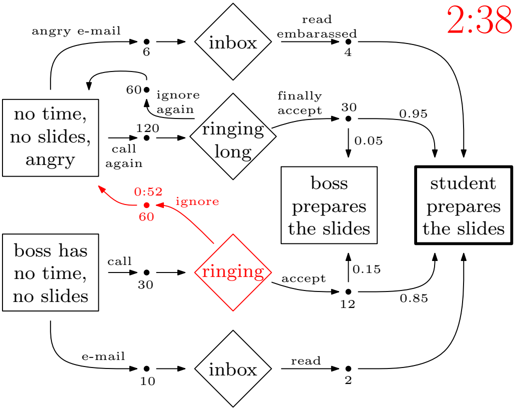


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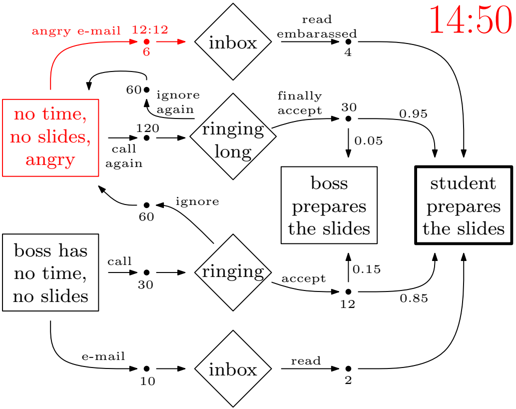


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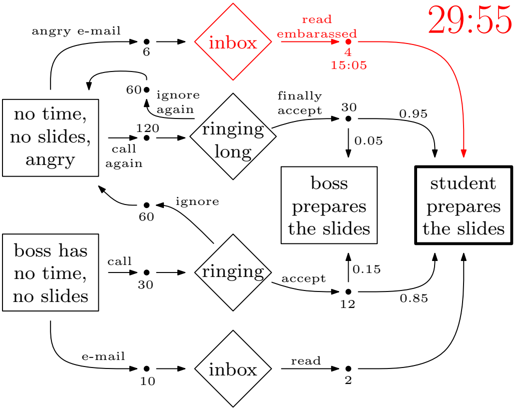


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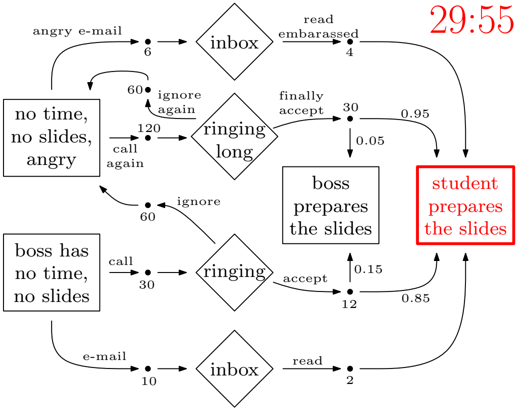


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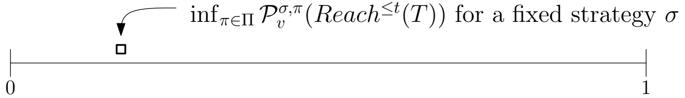

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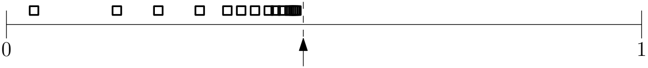


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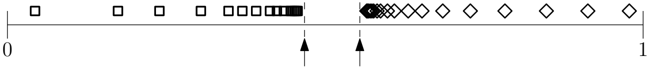


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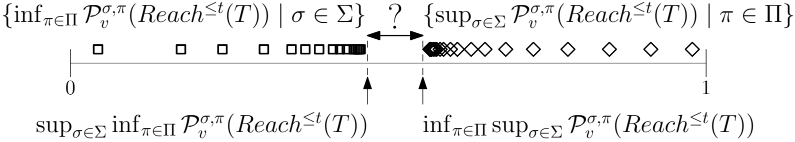


The game has a *value* in vertex $v$, denoted by $val(v)$, if

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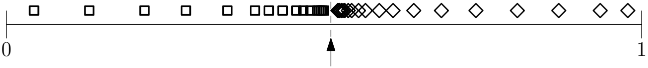


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## Theorem

*Every CTG G has a value in every vertex.*

# Optimal strategies

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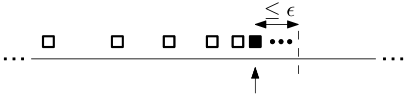
- Hence, there are $\epsilon$-optimal strategies!

for every $\epsilon > 0$ exists a strategy $\sigma_\epsilon$




$$\inf_{\pi \in \Pi} \mathcal{P}_v^{\sigma_\epsilon,\pi}(Reach^{\leq t}(T)) \geq val(v) - \epsilon$$

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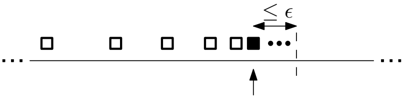


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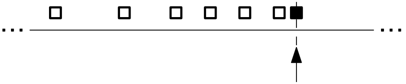
does there exist a fixed strategy $\sigma^*$ such that:

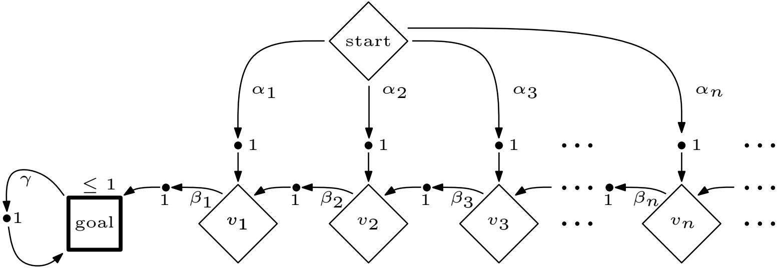


$$\inf_{\pi \in \Pi} \mathcal{P}_v^{\sigma^*, \pi}(Reach^{\leq t}(T)) = val(v)$$

- What about optimal strategies?

# Optimal strategies in CTGs

## Optimal strategies do not exist in general



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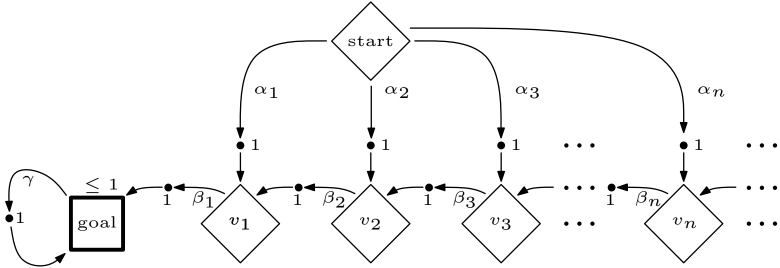


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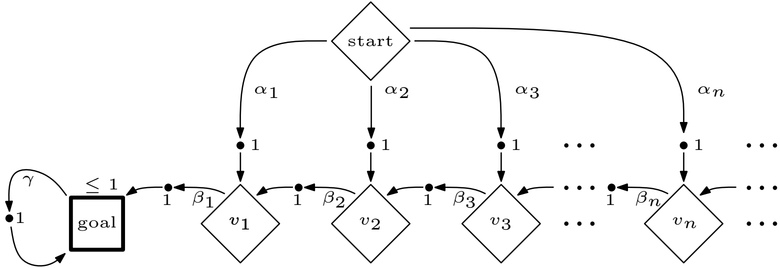


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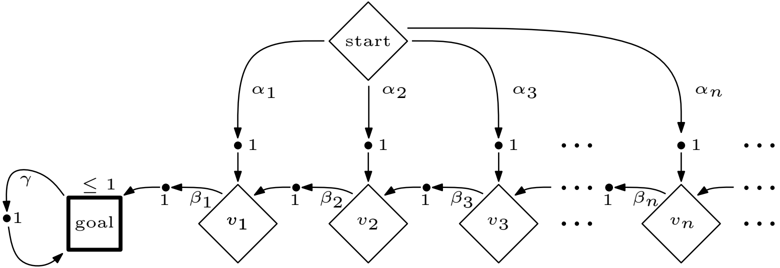


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### Theorem

*In every finitely-branching CTG $G$, the player $\Diamond$ has an optimal strategy.*

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- Strategies – functions with infinite domain (i.e. infinite set of histories)

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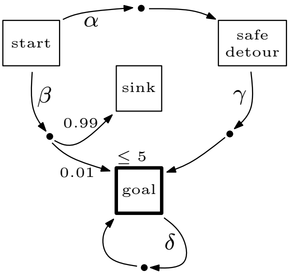

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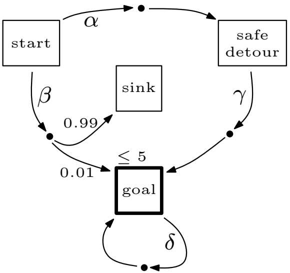

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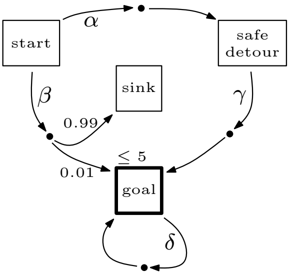


## Optimal strategy with finite description

- Up to the $k$-th step behaves like an optimal history-dependent strategy
- After the $k$-th step behaves greedily

# Algorithmical results

### Theorem

*For a finite CTG $G$ and $\epsilon > 0$, there are $\epsilon$-optimal strategies for both players computable in time*
$|V| \cdot |A| \cdot bp^2 \cdot \frac{1}{\epsilon}^{\mathcal{O}(1)} \cdot \left((max\mathcal{R}) \cdot t + \ln \frac{1}{\epsilon}\right)^{\mathcal{O}(|\mathcal{R}|)}$.

### Theorem

*For a finite uniform CTG $G$, the finite descriptions of optimal strategies for both players are effectively computable.*

Thank you for your attention!