

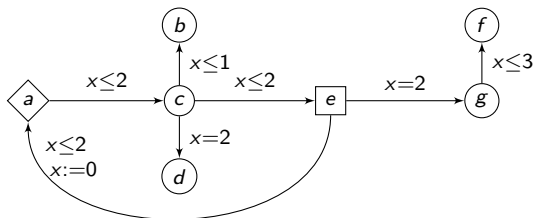
# Reachability in Stochastic Timed Games

Patricia Bouyer<sup>1</sup>   Vojtěch Forejt<sup>2</sup>

<sup>1</sup>LSV ENS Cachan, France

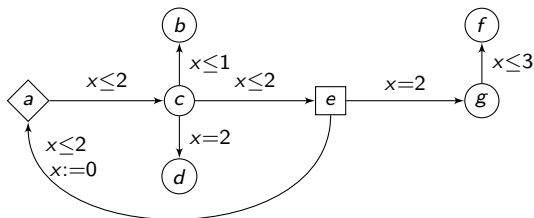
<sup>2</sup>Masaryk University, Czech Republic

## Stochastic Timed Game – Example



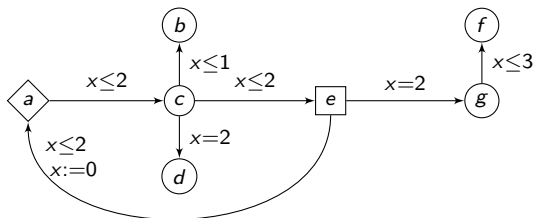
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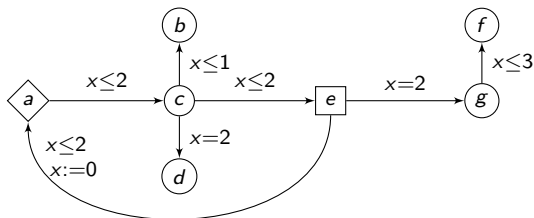
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  - Example: let  $x = 1.9$ , then the edge  $c \rightarrow b$  is not allowed

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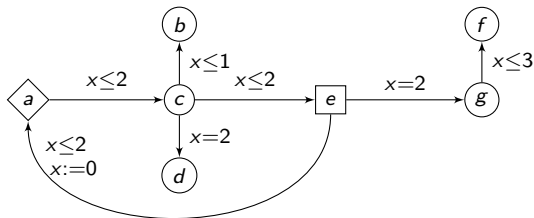
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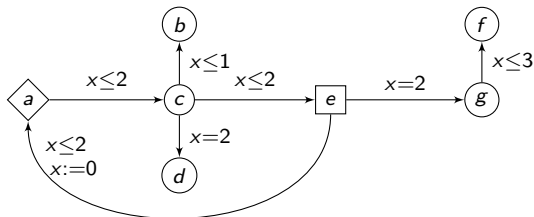
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  - Example: let  $x = 1.9$ , then the edge  $c \rightarrow b$  is not allowed
- Token is placed on the initial vertex, players choose when and along which edge it will move.
- Time elapses at uniform speed in every clock

## Stochastic Timed Game – Example cont.



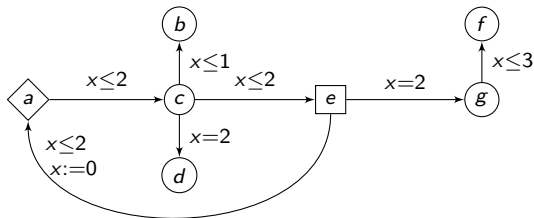
- Elpis and Ares choose their moves based on the history of the play.

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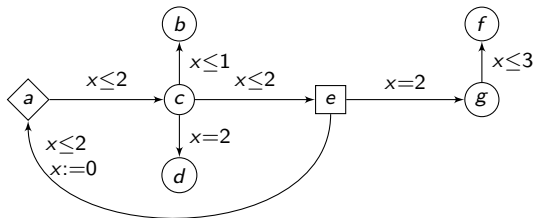
- Elpis and Ares choose their moves based on the history of the play.
- Tyche chooses based on the prescribed probability distribution.

# Probabilities



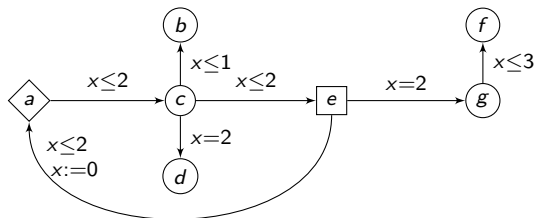
- State = vertex + clock values;
- For every state we have a set  $I \subseteq \mathbb{R}$  of delays after which some edge is enabled, and a distribution on  $I$ .
- Each edge has its weight  $w \in \mathbb{N}$

# Probabilities



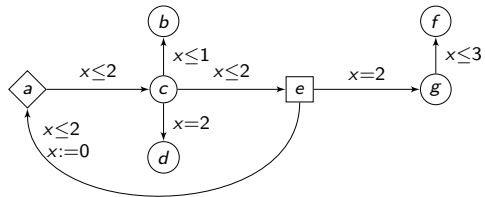
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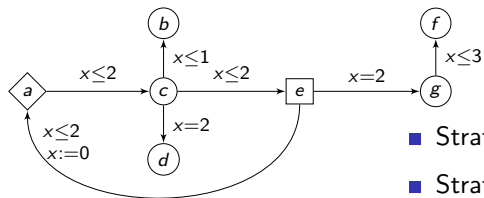


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- Example: for  $(c, 0.5)$  we have  $I = [0, 1.5]$ , with uniform distributions and weights 1, prob. of move to  $b$  is  $\frac{1}{6}$ .

# Play – Example

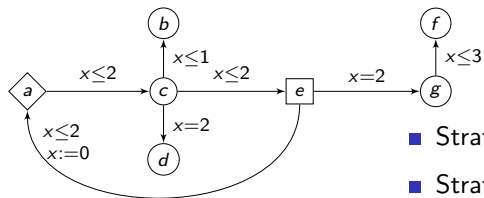


## Play – Example



- Strategy for Elpis: go to *c* when  $x = 1$
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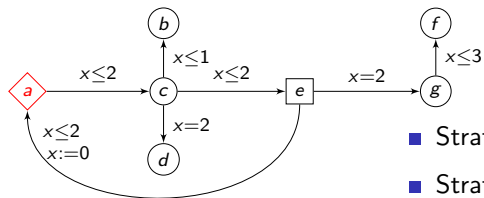
## Play – Example



- Strategy for Elpis: go to  $c$  when  $x = 1$
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- From the game and the strategy we obtain a Markov chain

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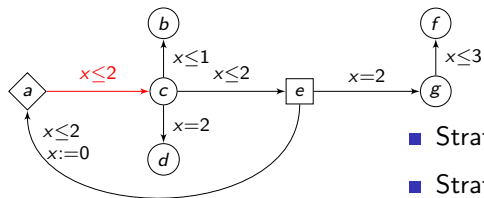


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$(a, 0)$

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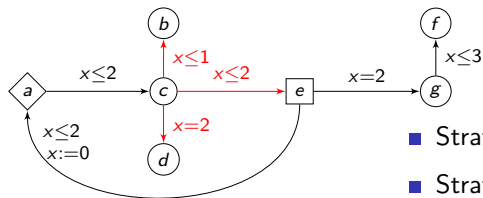


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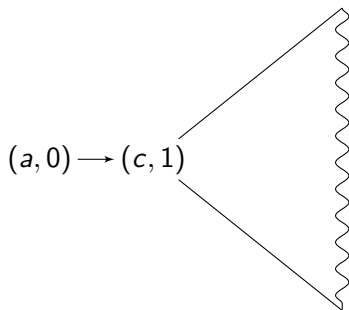
$$(a, 0) \rightarrow (c, 1)$$

## Play – Example

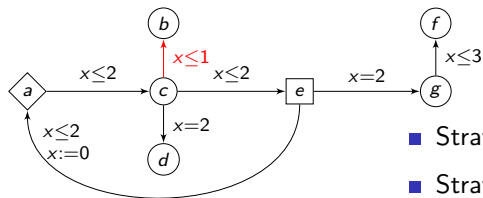


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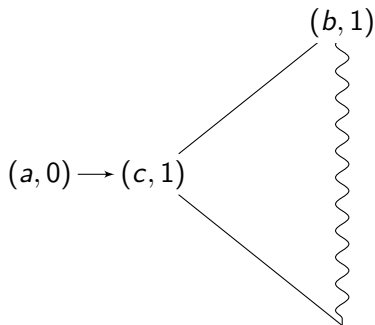


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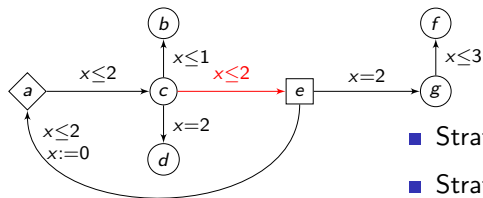


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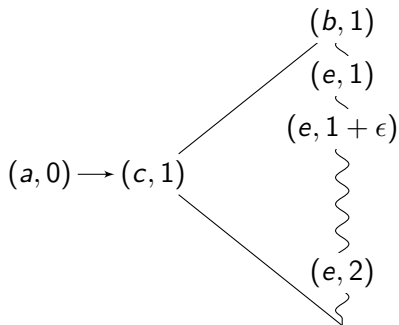


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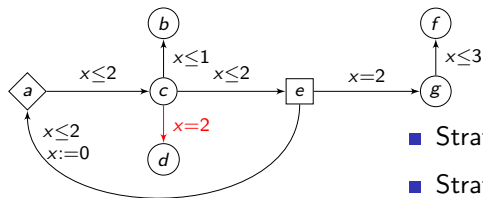


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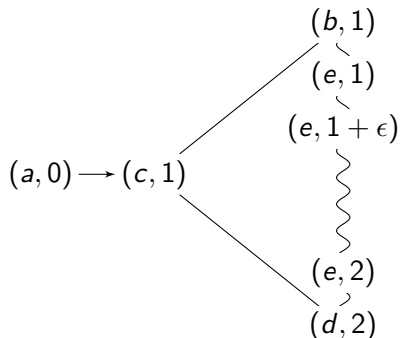


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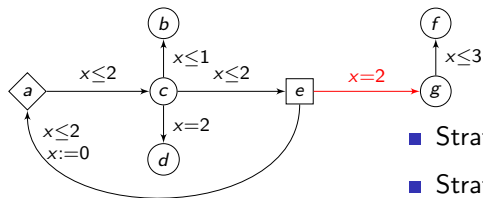


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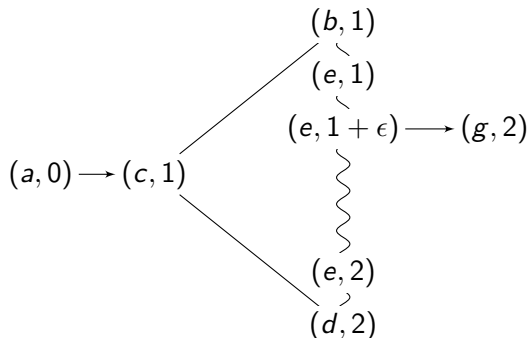


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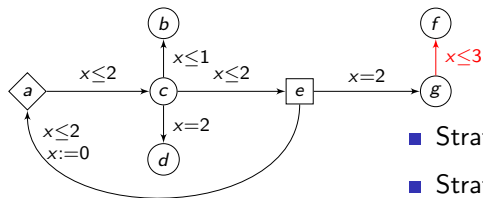


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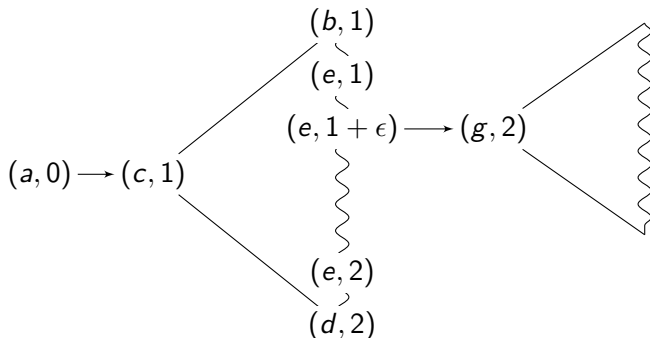


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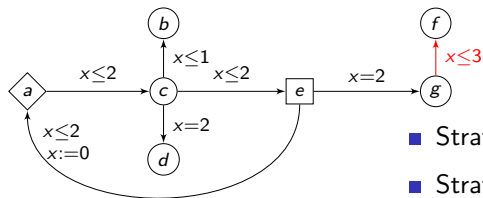


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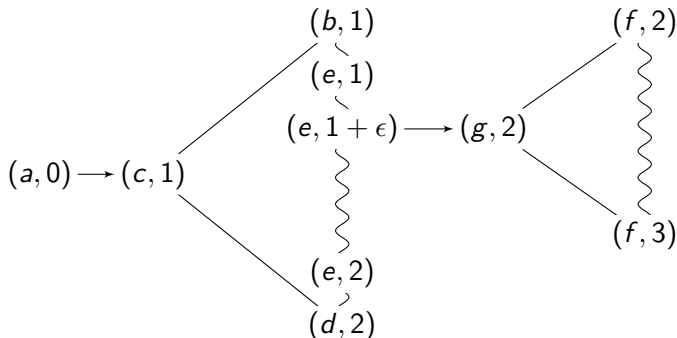


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# Motivation

- Extension of cont. time Markov decision processes
- Extension of timed automata

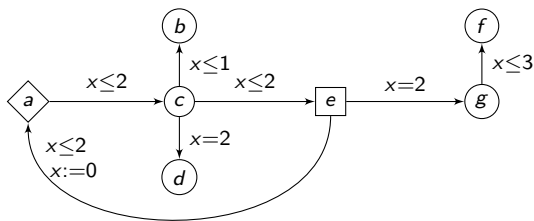
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- Extension of cont. time Markov decision processes
- Extension of timed automata
  
- Formal verification of systems
- Probability allows to model random behaviour of the system
- Clocks allow to model duration of events
- Elpis and Ares “control” the system

# The Reachability Problem

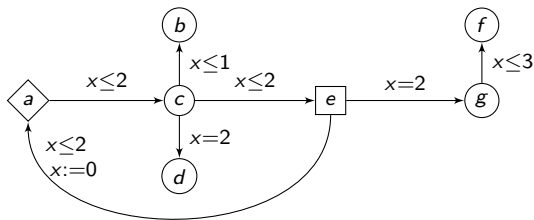
Given a game  $G$ , vertices  $V$  and a rational number  $r$ ,  
is there a strategy for Elpis s. t.  
for all strategies of Ares  
 $V$  is reached with pr. exactly (at most...)  $r$ ?

## Reachability Problem – Example



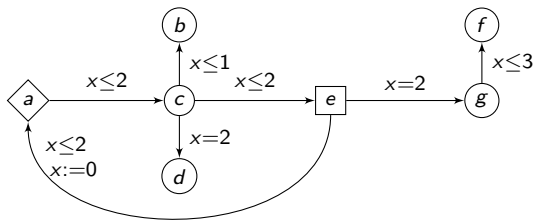
- Let us have uniform distributions
- Are vertices  $\{b, f\}$  reachable with prob. 1?

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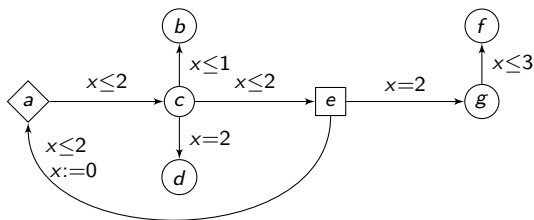
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  - No.

## Results – Undecidability

### Theorem

*The reachability problem for stochastic timed games is undecidable.*

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- Holds for uniform and exponential distributions on delays
- Proof by reduction from halting problem of two-counter machine to the reachability with prob.  $1/2$

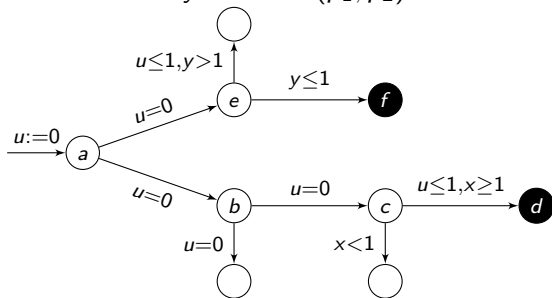
### Theorem

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- Elpis “simulates” a computation of the two-counter machine and “stores” counter values to clock values
- Elpis encodes  $p_1$  and  $p_2$  as  $1/(2^{p_1}3^{p_2})$
- Ares can check that Elpis is not cheating

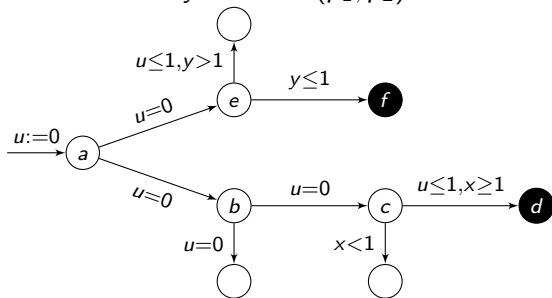
# Undecidability – Comparing Counter Values

- Check if clock  $y$  encodes  $(p_1, p_2)$  and  $x$  encodes  $(p_1 + 1, p_2)$



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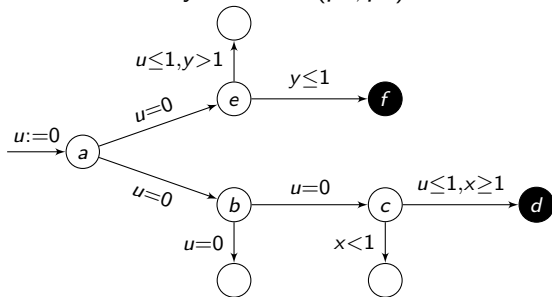
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- vertices  $d, f$  are reached with prob.  $\frac{1}{2}$  iff  $\frac{1}{2} = \frac{1}{4} \cdot x + \frac{1}{2} \cdot (1 - y)$   
 $y = 2x$

## Results – Decidability

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- Valid for all “reasonable” distributions on delays (e.g. uniform, exponential)
- Observation: Delays can be partitioned into finitely many intervals, in each of them a strategy may choose same behaviour independent of the precise time.
- Proof relies on the fact that a clock reset allows to forget everything about the past

## Future Work

- Reachability with prob.  $< 1$
- $\varepsilon$ -optimal strategies
- Time-bounded reachability
- Other winning objectives

# Conclusions

- We defined a new kind of games that incorporates time and probabilities in a natural way
- We proved that the (untimed) reachability problem for this model is
  - undecidable in general
  - decidable for  $1\frac{1}{2}$  player games if the probability to be reached is  $=1, >0, =0$
- Many interesting questions have yet to be answered.