

Characterizations of Branching Time Future Temporal Logics on Finite Trees

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Introduction

- **Verification** often uses **logic** to specify the expected behavior of systems.
- Determining the **expressive power** of these logics is an important question.
- In the case of strings, most known effective characterizations were obtained by the methods of **algebraic theory of automata and semigroups**.
- Another method to obtain inexpressibility results is provided by the **Ehrenfeucht-Fraïssé type games**.

We are concerned with the **definability problem** of some classes of logics on (finite, ranked, ordered) trees.

- Notation and the logic CTL
- The logics $\text{FTL}(\mathcal{L})$
- Algebraic characterization of $\text{FTL}(\mathcal{L})$
- Game-theoretic characterization of $\text{FTL}(\mathcal{L})$

Trees, tree languages

- A **rank type** R is a finite set of nonnegative integers, containing 0.
- $\Sigma = \bigcup_{n \in R} \Sigma_n$: finite **ranked alphabet** of rank type R .
- T_Σ stands for the set of (variable-free) Σ -trees.
- **Tree language** (over Σ): any set $L \subseteq T_\Sigma$.
- $t|_x$ stands for the **subtree of t rooted at the node x** .

Tree automata

- A (finite) Σ -tree automaton $\mathbb{A} = (A, \Sigma)$ consists of a (finite) state set A and an elementary operation $\sigma^{\mathbb{A}} : A^n \rightarrow A$ for each $\sigma \in \Sigma_n$.
- Given also a set $A' \subseteq A$ of accepting states, \mathbb{A} recognizes the language $L = \{t : t^{\mathbb{A}} \in A'\}$ with the set A' of accepting states.
- L is regular if it can be recognized by a finite tree automaton.
- \mathbb{A}_L denotes the minimal recognizer of L (unique up to isomorphism).

The logic CTL (Emerson, Clarke 1982)

Formulae of CTL over finite trees

- For any $\sigma \in \Sigma$, p_σ is a formula.
- Whenever φ and ψ are formulae, then so are $(\neg\varphi)$ and $(\varphi \vee \psi)$.
- If φ is a formula, $\text{EX}(\varphi)$ is also a formula.
- If φ and ψ are formulae, then $\text{EU}(\varphi, \psi)$ is a formula as well.

Semantics

A Σ -tree $t = \sigma(t_1, \dots, t_n)$ **satisfies** φ , in notation $t \models \varphi$ iff one of the following conditions hold:

- $\varphi = p_\sigma$;
- Boolean connectives are handled as usual;
- $\varphi = \text{EX}(\varphi_1)$ and $t_i \models \varphi_1$ for some $i \in [n]$;
- $\varphi = \text{EU}(\varphi_1, \varphi_2)$ and there exists a node $x \in \text{dom}(t)$ such that
 - $t|_x \models \varphi_2$ and
 - for any ancestor y of x , $t|_y \models \varphi_1$.

The logic CTL (Emerson, Clarke 1982)

A formula φ **defines** the language $\{t \in T_\Sigma : t \models \varphi\}$.

The definability problem of CTL

Given a regular tree language L (say, by \mathbb{A}_L), is L definable in CTL?

The decidability status of this problem is **unknown**.

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The logic $\text{FTL}(\mathcal{L})$ (Ésik 2005)

Syntax

For a class \mathcal{L} of tree languages, the formulae of $\text{FTL}(\mathcal{L})$ (over the alphabet Σ) are the following:

- for each $\sigma \in \Sigma$, p_σ is a formula;
- whenever φ and ψ are formulae, then so are $(\neg\varphi)$ and $(\varphi \vee \psi)$;
- if $L \in \mathcal{L}$ is a Δ -tree language and for each $\delta \in \Delta$, φ_δ is a formula, then $L(\delta \mapsto \varphi_\delta)$ is also a formula.

Semantics

For any tree $t \in T_\Sigma$ and formula φ , let $t \models \varphi$ iff

- $\varphi = p_\sigma$ and $\text{Root}(t) = \sigma$;
- Boolean connectives are treated as usual;
- $\varphi = L(\delta \mapsto \varphi_\delta)$ and the **characteristic tree** of t determined by the family $\{\varphi_\delta : \delta \in \Delta\}$ belongs to L .

Characteristic tree

The **characteristic tree** $\hat{t} \in T_\Delta$ of t determined by $\{\varphi_\delta : \delta \in \Delta\}$ is a relabeling of t :

For any n -ary node x , let $\hat{t}(x) = \delta$ iff

- $t|_x \models \varphi_\delta$ and δ is the first such element of Δ_n , **OR**
- $t|_x \not\models \bigvee_{\delta' \in \Delta_n} \varphi_{\delta'}$ and δ is the last element of Δ_n .

(We assume that each alphabet Δ comes with a fixed linear ordering. This has no impact on the expressive power.)

A formula φ over Σ **defines** the tree language $L_\varphi = \{t \in T_\Sigma : t \models \varphi\}$.

Definability problem

Many temporal logics, including CTL and the modular temporal logic can be expressed as $\text{FTL}(\mathcal{L})$ for some simple class \mathcal{L} of regular tree languages.

The definability problem of $\text{FTL}(\mathcal{L})$ for a class \mathcal{L} of (regular) tree languages

Given a (regular) tree language L , is L definable in $\text{FTL}(\mathcal{L})$?

- The **decidability status** of the problem for CTL is still unknown.
- Positive results were given for some **fragments** of CTL (Ésik '05, Bojańczyk and Walukiewicz '06, Ésik and Iván '08).

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Moore product of tree automata

Let $\mathbb{A} = (A, \Sigma)$ and $\mathbb{B} = (B, \Delta)$ be two tree automata and $\alpha : A \times \Sigma \rightarrow \Delta$ a rank-preserving function.

Then the **Moore product** of \mathbb{A} and \mathbb{B} determined by α is the tree automaton $\mathbb{A} \times_{\alpha} \mathbb{B} = (A \times B, \Sigma)$ with

$$\sigma^{\mathbb{A} \times_{\alpha} \mathbb{B}}((a_1, b_1), \dots, (a_n, b_n)) = (\sigma^{\mathbb{A}}(a_1, \dots, a_n), \delta^{\mathbb{B}}(b_1, \dots, b_n))$$

with $\delta = \alpha(\sigma^{\mathbb{A}}(a_1, \dots, a_n), \sigma)$ for each $n \in R$, $\sigma \in \Sigma_n$, $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$.

Theorem [Ésik, Iván 2008]

Let \mathcal{L} be a class of regular tree languages.

Then a tree language K is definable in $\text{FTL}(\mathcal{L})$ if and only if K is regular and \mathbb{A}_K is contained in the least class \mathbf{V} of finite tree automata satisfying each of the following conditions:

- \mathbf{V} contains \mathbb{A}_L for each $L \in \mathcal{L}$;
- \mathbf{V} contains a specific tree automaton \mathbb{D}_0 ;
- \mathbf{V} is closed under taking direct products, homomorphic images, subautomata and renamings (that is, a **pseudovariety** of finite tree automata);
- \mathbf{V} is closed under the **Moore product**.

(If each quotient of each language in L is definable in $\text{FTL}(\mathcal{L})$.)

- Notation and the logic CTL
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- **Game-theoretic characterization of $\text{FTL}(\mathcal{L})$**

A game on trees (Ésik, Iván 2008)

Let \mathcal{L} be a class of tree languages, $n \geq 0$ an integer and t_0, t_1 be trees over some alphabet Σ .

The n -round FTL(\mathcal{L})-game on (t_0, t_1) is played between Spoiler and Duplicator as follows:

- If $\text{Root}(t_0) \neq \text{Root}(t_1)$, Spoiler wins.
- Otherwise, if $n = 0$, Duplicator wins.
- Otherwise, Spoiler picks
 - a language $L \in \mathcal{L}$ over an alphabet Δ ;
 - Δ -relabelings \hat{t}_0 and \hat{t}_1 of t_0 and t_1 , resp., such that exactly one of them is in L .
- Duplicator picks two nodes, x of t_i and y of t_j ($i = j$ is allowed) with $\hat{t}_i(x) \neq \hat{t}_j(y)$.
- An $(n - 1)$ -round FTL(\mathcal{L})-game is played on $(t_i|_x, t_j|_y)$. Whoever wins the subgame, wins also the game.

Game-theoretic characterization of $\text{FTL}(\mathcal{L})$

Theorem [Ésik, Iván 2008]

For any class \mathcal{L} of tree languages, integer $n \geq 0$ and trees s, t over an alphabet Σ , the following are equivalent:

- 1 Duplicator has a winning strategy in the n -round $\text{FTL}(\mathcal{L})$ -game on (s, t) ;
- 2 s and t agree on all $\text{FTL}(\mathcal{L})$ -formulae over Σ with nesting depth at most n .

Corollary

Suppose \mathcal{L} is a **finite** class of tree languages. The following are equivalent for any Σ -tree language L :

- 1 L is definable in $\text{FTL}(\mathcal{L})$;
- 2 there exists an $n \geq 0$ such that Spoiler has a winning strategy in the n -round $\text{FTL}(\mathcal{L})$ -game over any pair (s, t) of Σ -trees with $s \in L$, $t \notin L$.

Application for $TL[EF^*]$

Syntax

The formulae of $TL[EF^*]$ over Σ are given as follows:

- for each $\sigma \in \Sigma$, p_σ is a formula;
- whenever φ and ψ are formulae, so are $(\neg\varphi)$ and $(\varphi \vee \psi)$;
- whenever φ is a formula, so is $EF^*(\varphi)$.

Semantics

The tree t satisfies $EF^*(\varphi)$ if $t|_x \models \varphi$ for some node x of t .

Observe that $EF^*(\varphi)$ is a shorthand for $EU(\top, \varphi)$, thus $TL[EF^*]$ is a syntactic fragment of CTL.

Characterization of $TL[EF^*]$

Theorem (for binary trees) [Ésik, Iván 2008]

Let $R = \{0, 2\}$. Then L is definable in $TL[EF^*]$ iff L is regular and \mathbb{A}_L satisfies the following identities:

- $\sigma(x, y) = \sigma(y, x)$;
- $\sigma(x, y) = \sigma(x, x)$, whenever $y \preceq x$;
- $\sigma(x, y) = \sigma(x', y')$, whenever $x \sim x'$ and $y \sim y'$;
- $\sigma(x, \sigma(x, y)) = \sigma(x, y)$.

Here

- $x \preceq y$ iff the state y is accessible from the state x ;
- $x \sim y$ iff $x \preceq y$ and $y \preceq x$.

Summary

- We defined the **Moore product** of tree automata providing an **algebraic characterization** of $\text{FTL}(\mathcal{L})$.
- We defined the **n -round $\text{FTL}(\mathcal{L})$ -game** providing a **game-theoretic characterization** of $\text{FTL}(\mathcal{L})$.
- We showed the **decidability** of the definability problem of some **fragments of CTL**, e.g. that of $TL[EF^*]$.

Thank you for your attention.