Noise and the Mermin-GHZ Game

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Outline

- Quantum information processing allows us to solve tasks that we cannot solve in the classical world.
- We are interested in the question whether quantum entanglement can be used to solve some distributed problems without any form of direct communication among the parties.
- The answer is negative if the result of computation must become known to at least one party.
- But if we are satisfied with establishment of nonlocal correlations among the parties' inputs and outputs, the answer is positive.

- A two party game $\mathcal{G}$ is a sextuple

$$\mathcal{G} = (X, Y, A, B, P, W)$$

where X, Y are input sets, A, B are output sets, P is a subset of $X \times Y$ known as a promise and $W \subseteq X \times Y \times A \times B$ is a winning condition.

Pseudo-Telepathy Games

Two party games II

- Before the game begins, Alice and Bob are allowed to discuss strategy and exchange any amount of classical information, including values of random variables.
- Alice and Bob may also share unlimited amount of entanglement.
- Afterwards, Alice and Bob are separated from each other and they are not allowed to communicate any more till the end of the game.

- Alice and Bob are given inputs $x \in X$ and $y \in Y$, respectively.
- Their task is to produce $a \in A$ and $b \in B$, respectively.
- The pairs (x, y) and (a, b) are called a question and an answer, respectively.
- Alice and Bob win the game if either $(x, y) \notin P$ or $(x, y, a, b) \in W$.
- A strategy of Alice and Bob is winning if it always allows them to win.

Pseudo-Telepathy Games

Definition (Brassard, Cleve, Tapp, 1999)

- We say that a two-party game is pseudo-telepathic if there is no winning strategy if Alice and Bob are restricted to be classical players, but there is a winning strategy, provided Alice and Bob share quantum entanglement.

Pseudo-Telepathy Games

A general quantum strategy

- Alice and Bob share an entangled quantum state $|\varphi\rangle$.
- After they have been given their inputs $x \in X$ and $y \in Y$, the players apply on their parts of $|\varphi\rangle$ unitary transformations U_x and U_y, respectively.
- Finally, the players perform measurements M_x and M_y on their parts of the shared state which give them their outputs $a \in A$ and $b \in B$, respectively.

- An experimental implementation of a quantum winning strategy is very hard to be perfect.
- Alice and Bob may perform imperfectly the unitary transformations required by the winning strategy.
- Moreover, they may not be able to keep the required entangled quantum state.
- Finally, their measurement devices may fail to give a correct outcome or may fail to give an outcome at all.

- Let G be any pseudo-telepathy game. We are interested in the question how good the quantum winning strategy for G is in the presence of quantum noise.
- The players are supposed to share a quantum state $E(|\varphi\rangle\langle\varphi|)$ where E is a superoperator which realizes some noisy quantum channel.

- After they have applied their unitary transformations, the players obtain a state

$$\rho_{err} = (U_x \otimes U_y)E(|\varphi\rangle\langle\varphi|)(U_x^\dagger \otimes U_y^\dagger).$$

- For a question (x, y), let $P_E(x, y)$ be a probability of the event that Alice and Bob obtain after measuring ρ_{err} a state corresponding to a correct answer to (x, y).
- We intend to compute a probability P_E of the event that Alice and Bob obtain after measuring ρ_{err} a state corresponding to a correct answer to a question which is chosen uniformly and randomly from P.

- Alice, Bob and Charles have each one bit as an input with the promise that the parity of the input bits is 0. We denote the input bits x_1, x_2 and x_3.
- The task for each of them is to produce one bit so that the parity of the output bits is equal to the disjunction of the input bits.
- Thus, if a_1, a_2 and a_3 are the outputs, then the equality $a_1 \oplus a_2 \oplus a_3 = x_1 \vee x_2 \vee x_3$ must hold.
- The best possible classical strategy for the Mermin-GHZ game succeeds with probability $0,75$.

Noise and the Mermin-GHZ Game

Quantum winning strategy

- Initially, Alice, Bob and Charles share the entangled state

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

- After the players have received their inputs x_1, x_2, x_3, each of them does the following:
  1. Applies to his or her register the unitary transformation U which is defined as

$$U(|0\rangle) = |0\rangle,$$

$$U(|1\rangle) = e^{\frac{\pi \imath x_i}{2}}|1\rangle.$$

  2. Applies the Hadamard transformation H to his or her register.
  3. Performs a measurement on his or her register in the computational basis. Outputs the bit a_i which he or she has measured.

- A quantum bit is with probability α replaced with the completely mixed state.
- It holds that

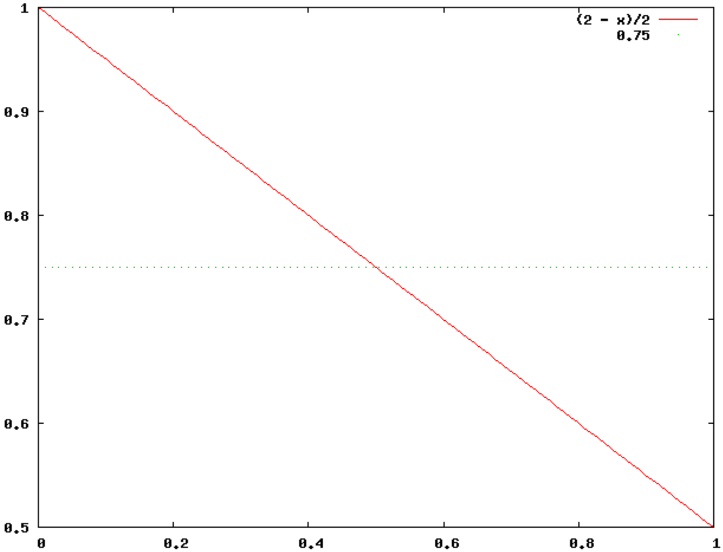
$$P_{\alpha}(x_1, x_2, x_3) = \frac{2-\alpha}{2}$$

for any question (x_1, x_2, x_3) satisfying the promise.
- It follows that

$$P_{\alpha} = \frac{2-\alpha}{2}.$$

- It is not very hard to see that quantum players are better than classical ones in the presence of the depolarizing channel if $\alpha < \frac{1}{2}$.

Depolarizing channel II



Noise and the Mermin-GHZ Game
Bit flip channel

- The state of a quantum bit is flipped from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability α.
- It holds that

$$P_\alpha(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 = x_2 = x_3 = 0 \\ 2\alpha^2 - 2\alpha + 1 & \text{otherwise.} \end{cases}$$

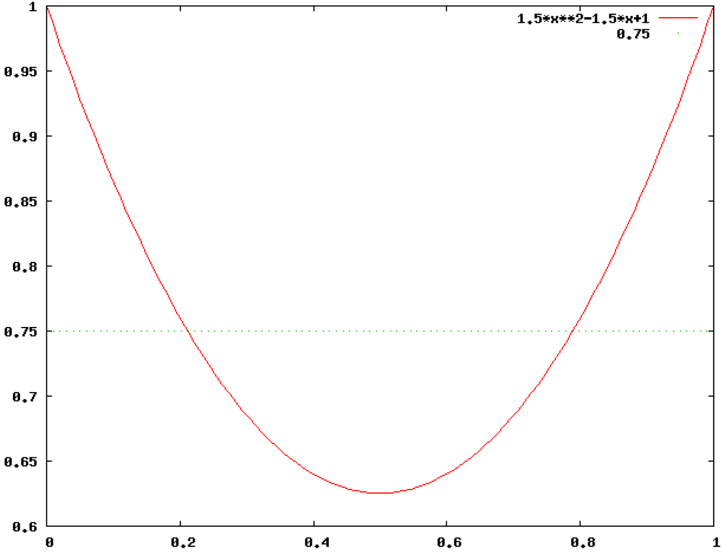
- It follows that

$$P_\alpha = \frac{1}{4} + \frac{3}{4}(2\alpha^2 - 2\alpha + 1) = \frac{3}{2}\alpha^2 - \frac{3}{2}\alpha + 1.$$

- Quantum players are better than classical ones in the presence of the bit-flip channel if $\alpha \in [0, \frac{3-\sqrt{3}}{6}) \cup (\frac{3+\sqrt{3}}{6}, 1]$.

Noise and the Mermin-GHZ Game

Bit flip channel II



Noise and the Mermin-GHZ Game

Phase flip channel

- The state $|1\rangle$ is transformed into $-|1\rangle$ (and vice versa) with probability α.
- It holds that

$$P_\alpha(x_1, x_2, x_3) = -4\alpha^3 + 6\alpha^2 - 3\alpha + 1$$

  for any inputs x_1, x_2 and x_3 satisfying the promise.
- I follows that

$$P_\alpha = -4\alpha^3 + 6\alpha^2 - 3\alpha + 1.$$

- Quantum players are better than classical ones in the presence of the bit-flip channel if $\alpha \in [0, -\sqrt[3]{\frac{1}{16}} + \frac{1}{2})$.

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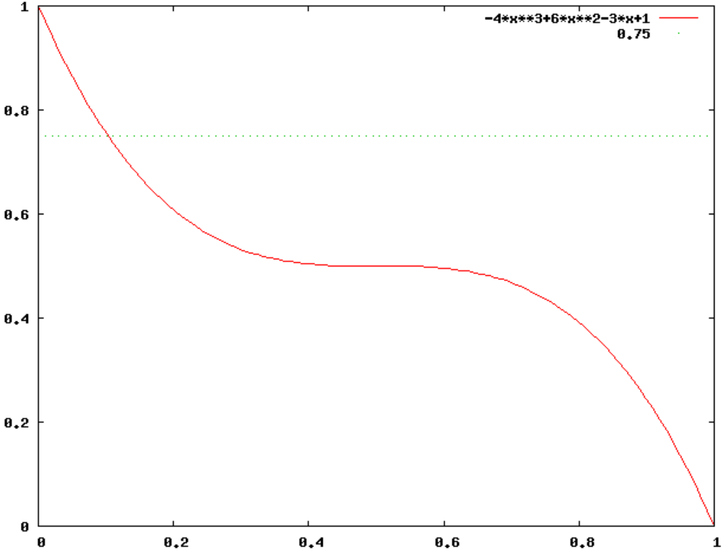
Phase flip channel II



Conclusions

- We have studied the impact of several basic noisy quantum channels on the quantum winning strategy for the Mermin-GHZ game.
- It has turned out that all the channels are able to decrease the success probability of quantum players so that they have no advantage over classical players, provided the noise is sufficiently strong.
- We have also investigated how strong the noise can be so that quantum players would still be better than classical ones.

Open Problems

- Is there a quantum strategy for the Mermin-GHZ game which is better than the winning strategy in the presence of quantum noise?
- What is the impact of quantum noise on Mermin's parity game and on the extended parity game?
- What is the impact of quantum noise on other pseudo-telepathy games?