# Regularity in PDA Games Revisited

Václav Brožek

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- recursive calls of subprocedures
- various subclasses trading expressivness for decidability (or the complexity of the proof of regularity ☺ )

- PDA: symbols $\{X\}$, states $\{p, r, s\}$ and rules $pX \longrightarrow pXX$, $pX \longrightarrow rX$, $rX \longrightarrow r\varepsilon$, $rX \longrightarrow s\varepsilon$, $sX \longrightarrow s\varepsilon$.

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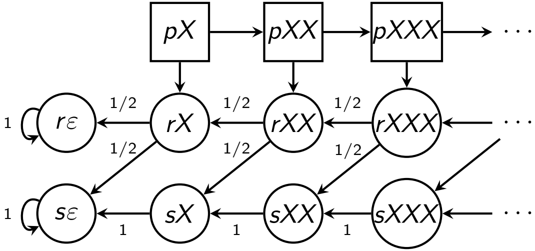

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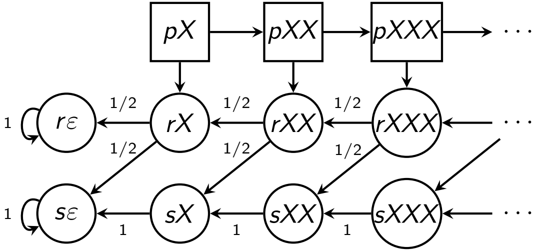
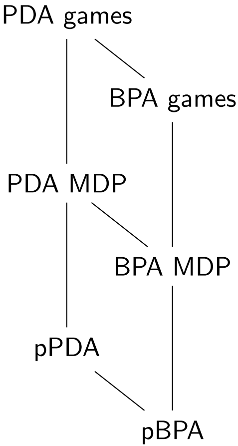



- strategy: history $\mapsto$ distribution over outgoing transitions,




games = 2 players + Nature

MDP = 1 player + Nature

p* = just Nature

BPA = 1 control state

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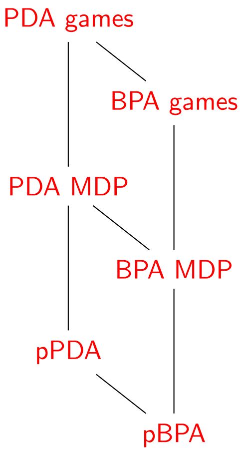
The objectives $({\leq}0)$, $({>}0)$, $({<}1)$ and $({\geq}1)$ are called *qualitative*.

The main question:

"Are the sets of winning configurations regular?"

# Known results

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eff. regular/regular/non-regular

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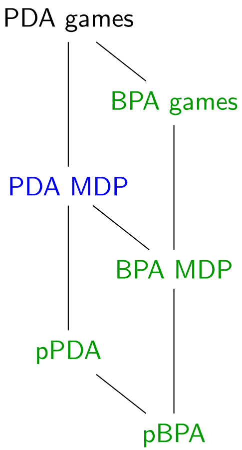


- general case: winning sets can be non-regular (even without players)
- qualitative case, PDA: regularity proved for at most 1 player
- qualitative case, BPA: regularity proved very recently

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- Proving regularity for winning with $T \subseteq \mathit{States} \times \{\varepsilon\}$ proves regularity for winning with any regular $T$.

# Our results

What remains to be solved?

Given a PDA game and a target set $T \subseteq \mathit{States} \times \{\varepsilon\}$, are the configurations where $\Box$ wins with probabilities $\geq 0$, $> 0$, $\geq 1$, $> 1$ regular?

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- The case $> 0$ reads: "Is there a path?"
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- The case $\geq 1$ (aka $= 1$) is interesting.

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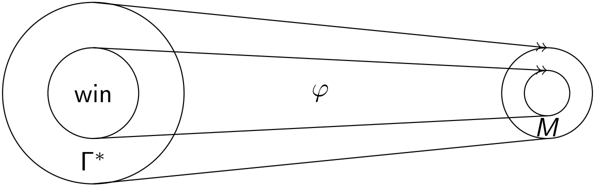


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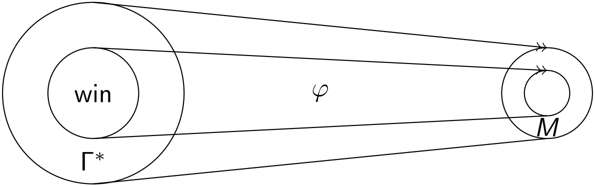


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For suitable $N_p \subseteq M$:

$$\Box \text{ wins exactly in } \bigcup_{p \in States} \{p\alpha \mid \alpha \in \varphi^{-1}(N_p)\}$$

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- $2^{n^2}$ in MDP without $\Box$

The bound $2^{n\cdot(2^n-1)}$ is tight for BPA ($n = 1$).

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A *value* of a configuration $p\alpha$ is

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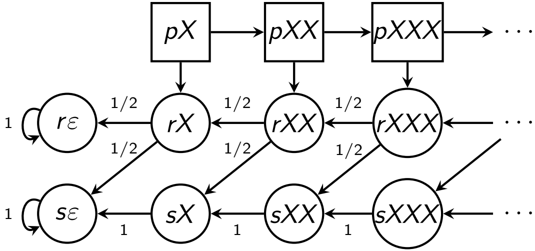
- $val^{-1}(0)$ is exactly where $\Diamond$ wins with $(\leq 0)$.
- $val^{-1}(1)$ is NOT where $\Box$ wins with $(\geq 1)$.
  However, using the technique as before, we can prove that $val^{-1}(1)$ is always regular.




$T = \{s\varepsilon\}$, $val(pX) = 1$, $\Box$ does not win ($\geq$1) in $pX$.

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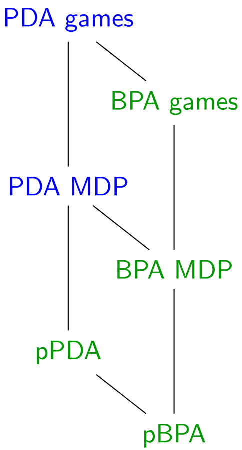


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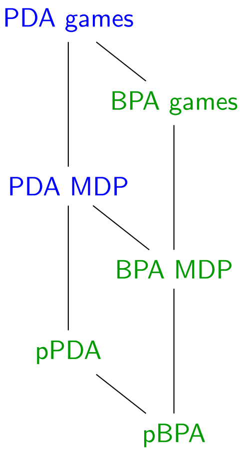


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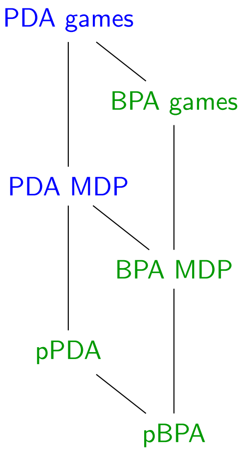


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- The upper bounds on representation were lowered for known cases.
- Regularity of the qualitative value shown as an application of the method.

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